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RESEARCH ARTICLE

A PDE approach for risk measures for derivatives with regime switching

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Abstract This paper considers a partial differential equation (PDE) approach to evaluate coherent risk measures for derivative instruments when the dynamics of the risky underlying asset are governed by a Markov-modulated geometric Brownian motion (GBM); that is, the appreciation rate and the volatility of the underlying risky asset switch over time according to the state of a continuous-time hidden Markov chain model which describes the state of an economy. The PDE approach provides market practitioners with a flexible and effective way to evaluate risk measures in the Markov-modulated Black–Scholes model. We shall derive the PDEs satisfied by the risk measures for European-style options, barrier options and American-style options.

Keywords Risk measures · Regime-switching PDE · Regime-switching HJB equation · Stochastic optimal control · Esscher transform · Delta-neutral hedging · Jump risk · American options · Exotic options

JEL Classification Numbers G32 · G13

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1 Introduction

Risk measurement and management are important areas in both the finance and insurance industries. Due to the proliferation of various modern financial and insurance products, there is a practical need to develop some appropriate methods and techniques to measure and manage the risk of these products. Some recent financial crises, such as the Asian financial crisis, the collapse of long-term capital management (LTCM), the turmoil at Barings and Orange Country, may be attributed to the inappropriate practice of risk measurement, especially the management of derivative instruments. This arouses the awareness of regulators of appropriate methods for risk measurement and the need for controling the riskiness of a trading portfolio. International regulatory bodies, such as Bank of International Settlement (BIS) and Basel Committee for Banking Supervision, have produced some reports with detailed guidelines on quantitative methods, techniques and practice for risk measurement and management. Regulators recommend the use of Value at Risk, VaR, for capital requirements while risk managers adopt VaR for allocating capital and imposing risk limits to different trading desks. Basically, VaR can be considered a statistical estimation of a portfolio's possible loss in such a way that the owner of the portfolio should be prepared to incur at least that loss with a given (small) probability level over a given (short) time horizon. See Morgan's (1996) Riskmetrics—technical document and Embrechts (2000) for an introduction, and Duffie and Pan (1997) for a survey. Duffie and Pan (1997) noted that VaR is not easy to implement if the portfolio contains derivatives. Since it is very difficult, if not impossible, to obtain a closed-form expression of VaR for complicated portfolios, market practitioners often resort to Monte Carlo simulation to approximate VaR for a portfolio of derivatives.

Although VaR is a popular tool for risk measurement in practice, it has been pointed out in some recent research studies that VaR does not satisfy some desirable theoretical properties. Artzner et al. (1999) developed an axiomatic approach to the construction of risk measures. They presented and justified a set of four desirable properties for risk measures. They introduced the class of coherent risk measures, which satisfy these four properties and obtained a representation for coherent risk measures. Subadditivity is one of the four desirable properties. Intuitively, it states that the merger of two risky positions should always reduce risk. Artzner et al. (1999) pointed out that VaR does not, in general, satisfy the subadditivity property, especially when the portfolio contains derivative instruments. This makes the search for an appropriate risk measure for derivative instruments of theoretical interest as well.

Measuring risks of portfolios containing derivative instruments is a challenging task. Much of the literature focuses on investigating VaR for portfolios of derivative instruments (see J.P. Morgan's Riskmetrics–technical document, Duffie and Pan (1997) and Jahel et al. (1999), etc). There are two common analytical approaches to the calculation of VaR for derivative portfolios, namely the delta-gamma approximation and the method of matching portfolios' moments. In the former method, the portfolio can be decomposed in terms of the portfolio's delta and gamma which are then used to approximate the portfolio's return in the calculation of the portfolio's distribution. In the latter method, one needs to calculate the portfolio's moments and find a distribution that matches the portfolio's distribution as closely as possible. Both approaches are difficult to implement when the portfolio contains complex derivatives or a significant number of derivatives.

Recently, there has been considerable interest in the applications of regime switching, or Markov-modulated, models to various financial problems. Some papers on the use of regime-switching models in finance include Elliott and van der Hoek (1997) for asset allocation, Pliska (1997) and Elliott et al. (2001) for short rate models, Elliott and Hinz (2002) for portfolio analysis and chart analysis, Guo (2001) and Chan et al. (2005) for option pricing under market incompleteness, Buffington and Elliott (2002a,b) for pricing European and American options, Elliott et al. (2003) for volatility estimation. There is a relatively little work on investigating quantitative risk measurement for financial instruments, especially derivative markets, in the context of regime-switching models.

This paper considers a partial differential equation (PDE) approach for evaluating coherent risk measures for derivative instruments in a Markov-modulated Black-Scholes environment. We consider the coherent risk measures for derivative instruments adopted in Siu and Yang (2000) and Siu and Yang (2001). We suppose that the dynamics of risky underlying asset are governed by a Markov-modulated geometric Brownian motion (GBM); that is, the appreciation rate and the volatility in the log-normal dynamics of the underlying risky asset depend on the states of the economy which are modelled by a continuous-time hidden Markov chain. In practice, one may use some observable economic indicators, such as GDP, the inflation rate or other observable economic factors, as a proxy for the hidden state of the economy. One may then estimate the hidden state of the economy using historical data from one, (or more), of these indicators. In particular, one may first specify a set of threshold values and divide the state space for the values of an economic indicator into different regimes. Then, the state of the hidden Markov chain can be estimated by the regime in which the value of the economic indicator falls. The transitions of the states can then be traced by the jumps in the historical data series of this indicator. Another way to deal with estimation the hidden state is to use filtering. However, the filtering problem for our case poses additional challenge since both the drift and the volatility of the log-normal dynamics switch over time according to the state of the hidden Markov chain. Some works on using filtering methods in finance include Elliott et al. (2003) for the log-normal dynamics with Markov-switching volatility and Elliott and Royal (2007) for regime switching variance Gamma dynamics. We also assume that the market interest rate of a bank account is a constant. We employ the regime-switching Esscher transform introduced in Chan et al. (2005) to select an equivalent martingale measure for pricing the options in the incomplete market setting. The concept of the Esscher transform, in particular the random Esscher transform, has been adopted in Siu et al. (2001) for pricing derivatives in the context of risk measurement under the

Bayesian framework. In addition to pricing, we also consider the hedging of derivatives in an incomplete market setting by considering delta-neutral hedging and evaluating the risk of incomplete hedging. As in Siu and Yang (2000, 2001), we adopt real-world probability measures for risk measurement. The partial differential equation (PDE) approach provides market practitioners with a flexible and effective way to evaluate risk measures in a Markov-modulated Black-Scholes model. It is especially useful for the case of American options, in which the possibility of early exercise needs to be taken into consideration in both the pricing and risk measurement parts. We derive the PDEs satisfied by the risk measures for European-style options and American-style options. We also demonstrate the use of the PDE approach for evaluating risk measures for exotic options, like barrier options.

The paper is organized in the following way. Section 2 describes the asset price dynamics of the Markov-modulated Black-Scholes environment and the marking the option prices in the market using the regime-switching Esscher transform in Chan et al. (2005). We use delta-neutral hedging for an option in the incomplete market described by the Markov-modulated Black–Scholes model. Section 3 defines a dynamic regime-switching version of the generalized "scenarios" expectation (GSE) for an option position, which is a coherent risk measure. We shall derive the regime-switching PDEs, satisfied by the GSE, and the corresponding system of N-coupled PDEs. We shall also derive a system of coupled PDEs for the GSE for the jump risk due to the hidden Markov chain. Section 4 derives the PDEs satisfied by the risk measure of a barrier option. We deal with the risk measure for an American-style option in Sect. 5. The final section summarizes the paper and suggests some possible topics for further investigation.

2 Standard European-style options

In this section, we shall describe the asset price dynamics and discuss the valuation of standard European-style options using the regime-switching Esscher transform described in Chan et al. (2005) in the context of a Markov-modulated Black–Scholes environment. We also discuss the delta-neutral hedging in this context. First, we consider a financial model consisting of one risk-free asset, namely a money market account, and one risky asset, namely stock or share, with the time horizon $[0, T]$ during which all economic activities take place. The risk-free asset and the risky asset are tradable continuously over time. We present the asset price dynamics of the model in the sequel.

Suppose $(\Omega, \mathcal{F}, \mathcal{P})$ is a complete probability space, where $\mathcal{P}$ is a reference physical probability measure. Let $\mathcal{T}$ denote the time index set $[0, T]$ of the model. We consider a continuous-time hidden Markov chain process $\{X_t\}_{t\in\mathcal{T}}$ on $(\Omega, \mathcal{F}, \mathcal{P})$ with a finite state space $\mathcal{X} := (x_1, x_2, \ldots, x_N)$. Without loss of generality, we can identify the state space of $\{X_t\}_{t\in\mathcal{T}}$ with a finite set of unit vectors $\{e_1, e_2, \ldots, e_N\}$, where $e_i = (0, \ldots, 1, \ldots, 0) \in \mathcal{R}^N$.

Write $A = [a_{ij}]_{i,j=1,2,\ldots,N}$ for the rate matrix of the Markov chain. Then, from Aggoun et al. (1994) (see Chap. 7 therein), we have the following semi-martingale representation decomposition for $\{X_t\}_{t\in\mathcal{T}}$:

$$X_t = X_0 + \int_0^t AX_s \mathrm{d}s + M_t, \tag{2.1}$$

where $\{M_t\}_{t\in\mathcal{T}}$ is an $\mathcal{R}^N$-valued martingale process with respect to the filtration generated by $\{X_t\}_{t\in\mathcal{T}}$.

We shall assume that the market interest rate of the bank account is a constant r. Then, the dynamics of the price process $\{B_t\}_{t\in\mathcal{T}}$ for the bank account are described by:

$$\mathrm{d}B_t = rB_t\mathrm{d}t, \quad B_0 = 1. \tag{2.2}$$

We suppose that the stock appreciation rate $\{\mu_t\}_{t\in\mathcal{T}}$ and the volatility $\{\sigma_t\}_{t\in\mathcal{T}}$ of the stock price process also depend on $\{X_t\}_{t\in\mathcal{T}}$ and are described by:

$$\mu_t := \mu(t, X_t) = \langle \mu, X_t \rangle, \quad \sigma_t := \sigma(t, X_t) = \langle \sigma, X_t \rangle, \tag{2.3}$$

where $\mu := (\mu_1, \mu_2, \ldots, \mu_N)$ and $\sigma := (\sigma_1, \sigma_2, \ldots, \sigma_N)$ with $\sigma_i > 0$ for each $i = 1, 2, \ldots, N$.

We suppose the dynamics of the stock price process $\{S_t\}_{t\in\mathcal{T}}$ are then given by the following Markov-modulated geometric Brownian motion:

$$\mathrm{d}S_t = \mu_t S_t \mathrm{d}t + \sigma_t S_t \mathrm{d}W_t, \quad S_0 = s, \tag{2.4}$$

where $W := \{W_t\}$ is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathcal{P})$ with respect to the filtration $\{\mathcal{F}_t^W\}$, which is the $\mathcal{P}$-augmentation of the natural filtration generated by W. We suppose that $\{X_t\}_{t\in\mathcal{T}}$ and $\{W_t\}_{t\in\mathcal{T}}$ are independent.

Let $\{\mathcal{F}_t^X\}_{t\in\mathcal{T}}$ denote the $\mathcal{P}$-augmentation of the natural filtration generated by $\{X_t\}_{t\in\mathcal{T}}$. For each $t \in \mathcal{T}$, we define $\mathcal{G}_t$ as the σ-algebra $\mathcal{F}_t^X \vee \mathcal{F}_t^W$. Now, we consider a generic standard European option V written on the underlying stock and with expiry time T. That is, V is a $\mathcal{G}_T$ measurable integrable random variable. We shall determine the price of the option V. Due to the additional uncertainty described by regime switching, the Markov-modulated Black-Scholes market is incomplete. Hence, there are infinitely many equivalent martingale measures for pricing options. Chan et al. (2005) introduced the regime-switching Esscher transform to determine an equivalent martingale pricing measure. We employ the pricing methodology in Chan et al. (2005) to determine this equivalent martingale pricing measure.

Let $\theta_t := \theta(t, X_t)$ denote the regime switching Esscher parameter, which can be written as follows:

$$\theta(t, X_t) = \langle \theta, X_t \rangle, \tag{2.5}$$

where $\theta := (\theta_1, \theta_2, \ldots, \theta_N) \in \mathcal{R}^N$. Then, the regime switching Esscher transform $\mathcal{Q}_\theta \sim \mathcal{P}$ on $\mathcal{G}_t$ with respect to a family of parameters $\{\theta_s\}_{s\in[0,t]}$ is given by:

$$\left.\frac{d\mathcal{Q}_\theta}{d\mathcal{P}}\right|_{\mathcal{G}_t} = \frac{\exp\left(\int\limits_0^t \theta_s \mathrm{d}W_s\right)}{E_{\mathcal{P}}\left[\exp\left(\int\limits_0^t \theta_s \mathrm{d}W_s\right)\Big|\mathcal{F}_t^X\right]}, \quad t \in \mathcal{T}. \tag{2.6}$$

Let $\{\tilde{\theta}_t\}_{t\in\mathcal{T}}$ denote a time-indexed family of risk-neutral regime switching Esscher parameters. Then, as in Chan et al. (2005), the martingale condition is given by considering an enlarged filtration as follows:

$$S_0 = E_{\mathcal{Q}_{\tilde{\theta}}}(\mathrm{e}^{-rt} S_t | \mathcal{F}_t^X), \quad \text{for any } t \in \mathcal{T}. \tag{2.7}$$

If $\mathrm{e}^{-rt} S_t$ is to be a martingale under $\mathcal{Q}_{\tilde{\theta}}$, it is shown in Chan et al. (2005) that $\tilde{\theta}_t = \frac{r-\mu_t}{\sigma_t}$ and the Radon–Nikodym derivative of $\mathcal{Q}_{\tilde{\theta}}$ is given by:

$$\left.\frac{\mathrm{d}\mathcal{Q}_{\tilde{\theta}}}{\mathrm{d}\mathcal{P}}\right|_{\mathcal{G}_t} = \exp\left[\int\limits_0^t \left(\frac{r-\mu_s}{\sigma_s}\right)\mathrm{d}W_s - \frac{1}{2}\int\limits_0^t \left(\frac{r-\mu_s}{\sigma_s}\right)^2 \mathrm{d}s\right]. \tag{2.8}$$

Then, using Girsanov's theorem, $\tilde{W}_t = W_t - \int\limits_0^t (\frac{r-\mu_s}{\sigma_s})\mathrm{d}s$ is a standard Brownian motion with respect to $\{\mathcal{G}_t\}_{t\in\mathcal{T}}$ under $\mathcal{Q}_{\tilde{\theta}}$. Hence, we can write the stock price dynamics under $\mathcal{Q}_{\tilde{\theta}}$ as:

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma_t S_t \mathrm{d}\tilde{W}_t. \tag{2.9}$$

Let $\tilde{\mathcal{G}}_t$ denote the σ-field $\mathcal{F}_t^W \vee \mathcal{F}_T^X$. Then, given $\tilde{\mathcal{G}}_t$, the price of the option V with payoff $V(S_T)$ at the expiry time T is given by:

$$V(t, T|\tilde{\mathcal{G}}_t) = E_{\mathcal{Q}_{\tilde{\theta}}}(\mathrm{e}^{-r(T-t)} V(S_T)|\tilde{\mathcal{G}}_t). \tag{2.10}$$

We shall derive a partial differential equation (PDE) for the option V under the Markov-modulated GBM. Following Buffington and Elliott (2002a,b), the price of the option V given $\mathcal{G}_t$ is given by:

$$V(t, T|\mathcal{G}_t) = E_{\mathcal{Q}_{\tilde{\theta}}}(V(t, T|\tilde{\mathcal{G}}_t)|\mathcal{G}_t). \tag{2.11}$$

Since (X, S) is a Markov process, given $S_t = s$ and $X_t = x$,

$$V(t, T, S, X) = E_{\mathcal{Q}_{\tilde{\theta}}}(\mathrm{e}^{-r(T-t)} V(S_T)|S_t = s, X_t = x). \tag{2.12}$$

Write $\tilde{V}(t, T, S, X)$ for $e^{-rt}V(t, T, S, X)$. Then

$$\tilde{V}(t, T, S, X) = E_{\mathcal{Q}_{\tilde{\theta}}}(e^{-rT}V(S_T)|\mathcal{G}_t). \tag{2.13}$$

Write $\tilde{\mathbf{V}}(t, T, S) := (\tilde{V}(t, T, S, e_1), \tilde{V}(t, T, S, e_2), \ldots, \tilde{V}(t, T, S, e_N))$. Then,

$$\tilde{V}(t, T, S_t, X_t) = \left\langle \tilde{\mathbf{V}}(t, T, S_t), X_t \right\rangle. \tag{2.14}$$

As in Buffington and Elliott (2002a,b), using Itô's differentiation rule and the fact that $\tilde{V}$ is a $(\mathcal{G}, \mathcal{Q}_{\tilde{\theta}})$-martingale, $\tilde{V}(t, T, S_t, X_t)$ satisfies the following regime switching P.D.E.

$$\frac{\partial \tilde{V}}{\partial t} + rS\frac{\partial \tilde{V}}{\partial S} + \frac{1}{2}\sigma_t^2 S^2 \frac{\partial^2 \tilde{V}}{\partial S^2} + \left\langle \tilde{\mathbf{V}}, AX \right\rangle = 0. \tag{2.15}$$

Write $\mathbf{V}(t, T, S) := (V(t, T, S, e_1), V(t, T, S, e_2), \ldots, V(t, T, S, e_N))$ so that $V(t, T, S, X) = \langle \mathbf{V}, X \rangle$. Since $\tilde{V}(t, T, S, X) := e^{-rt}V(t, T, S, X)$ we deduce from (2.15) that

$$-rV + \frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma_t^2 S^2 \frac{\partial^2 V}{\partial S^2} + \langle \mathbf{V}, AX \rangle = 0, \tag{2.16}$$

with terminal condition:

$$V(T, T, S, X) = V(S_T). \tag{2.17}$$

Here, we obtain a simple regime-switching PDE (2.16) for derivative pricing. The reason why the PDE is obtained is due to the assumption that the standard Brownian motion driving the stochastic differential equation for the price dynamics of the underlying risky asset and the hidden Markov chain are independent. Under this assumption, the martingale measure we use for pricing corresponds to the logarithmic utility for wealth in the context of "neutral derivative pricing" (see Kallsen (2000)). In this case, the risk premium is zero and the risk due to the switching of the hidden Markov chain is not priced. Stojanovic (2005) pointed out that this is the only obvious choice for the risk premium. However, in reality, actual investing behavior is less aggressive than that described by the logarithmic utility of wealth. Since the risk premium is zero, the stock appreciation rate does not affect the pricing PDE (2.16). The inputs for the pricing PDE (2.16) are the interest rate, the volatility and the transition probability matrix of the hidden Markov chain. The transition probability matrix affects the pricing PDE (2.16) only through the volatility, and not through the stock appreciation rate.

Therefore, the components $V_i = V(t, T, S, e_i)$ satisfy the following system of N-coupled partial differential equations:

$$-rV_i + \frac{\partial V_i}{\partial t} + rS\frac{\partial V_i}{\partial S} + \frac{1}{2}\sigma_i^2 S^2 \frac{\partial^2 V_i}{\partial S^2} + \langle \mathbf{V}, Ae_i \rangle = 0, \tag{2.18}$$

with the terminal condition:

$$V(T, T, S, e_i) = V(S_T), \quad i = 1, 2, \ldots, N. \tag{2.19}$$

In the sequel, we shall consider the hedging of the option under the Markov-modulated Black–Scholes framework. Here, we employ delta-neutral hedging for the short position of the option. First, for each $t \in \mathcal{T}$, let $\phi_0(t)$ and $\phi_1(t)$ denote two $\mathcal{G}_t$-predictable random variables, which represent the numbers of units of the bank account and the underlying risky asset at time t in a portfolio. Let Π_t denote the value of the hedged portfolio at time t. Then,

$$\Pi_t = \phi_0(t)B_t + \phi_1(t)S_t. \tag{2.20}$$

We suppose that Π is self-financing. That is,

$$\mathrm{d}\Pi_t = \phi_0(t)\mathrm{d}B_t + \phi_1(t)dS_t. \tag{2.21}$$

If we combine the hedged portfolio with a short position in the option, the overall hedged position has value:

$$\tilde{\Pi}(t, S, X) = -V(t, T, S, X) + \Pi_t. \tag{2.22}$$

Note that by Itô's differentiation rule,

$$\begin{aligned} \mathrm{d}V(t, T, S, X) = &\left(\frac{\partial V}{\partial t} + \mu_t S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma_t^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)\mathrm{d}t \\ &+\sigma_t S\frac{\partial V}{\partial S}\mathrm{d}W_t + \langle \mathbf{V}, \mathrm{d}X_t \rangle . \end{aligned} \tag{2.23}$$

Let $\mathcal{L}(V, t) := \frac{\partial V}{\partial t} + \mu_t S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma_t^2 S^2 \frac{\partial^2 V}{\partial S^2}$. Then,

$$\begin{aligned} &\mathrm{d}\tilde{\Pi}(t, S, X) \\ &= (-\mathcal{L}(V, t) + \phi_0 rB + \phi_1 \mu_t S - \langle \mathbf{V}, AX \rangle)\mathrm{d}t + \left(\phi_1 \sigma_t S - \sigma_t S\frac{\partial V}{\partial S}\right)\mathrm{d}W_t \\ &\quad - \langle \mathbf{V}, \mathrm{d}M_t \rangle . \end{aligned} \tag{2.24}$$

Suppose the portfolio is delta neutral. That is, $\frac{\partial \tilde{\Pi}}{\partial S} = 0$. Then,

$$\phi_1 \sigma_t S - \sigma_t S\frac{\partial V}{\partial S} = 0. \tag{2.25}$$

Hence,

$$\phi_1 = \frac{\partial V}{\partial S}, \quad \phi_0 = B^{-1}\left(\Pi_t - \frac{\partial V}{\partial S}S\right). \tag{2.26}$$

Under the delta-neutrality, the dynamics of $\tilde{\Pi}$ are given by:

$$\mathrm{d}\tilde{\Pi} = (-\mathcal{L}(V,t) + \phi_0 rB + \phi_1 \mu_t S - \langle \mathbf{V}, AX\rangle)\mathrm{d}t - \langle \mathbf{V}, \mathrm{d}M_t\rangle\,. \tag{2.27}$$

Note that

$$\mathcal{L}(V,t) + \langle \mathbf{V}, AX\rangle = rV. \tag{2.28}$$

Then,

$$\mathrm{d}\tilde{\Pi} = r\tilde{\Pi}\mathrm{d}t + \phi_1(\mu_t - r)S\mathrm{d}t - \langle \mathbf{V}, dM_t\rangle \tag{2.29}$$

The delta hedging method cannot eliminate the risk completely. In particular, the jump risk due to the hidden Markov chain cannot be hedged. The instantaneous jump risk is given by:

$$\phi_1(\mu_t - r)S\mathrm{d}t - \langle \mathbf{V}, \mathrm{d}M_t\rangle\,. \tag{2.30}$$

We shall discuss how to evaluate the jump risk due to the hidden Markov chain by the GSE in the next section.

3 Coherent risk measure: generalized "scenarios" expectation (GSE)

In this section, we shall derive the PDEs governing the dynamics of a coherent risk measure for an unhedged position of standard European options, such as calls, puts or digital options, over a fixed time horizon $[t, t+h]$ in the context of a Markov-modulated Black–Scholes market. We shall consider a coherent risk measure, namely the generalized scenarios expectation (GSE), for the option V over the time horizon $[t, t+h]$. This can be considered a general form of a (conditional) coherent risk measure first proposed by Artzner et al. (1999). We suppose that the option will be held during the time interval $[t, t+h]$. From a buyer's point of view, given the market information up to time u, $\mathcal{G}_u$, where $u \in [t, t+h)$, we define the GSE for an unhedged position of derivatives over $[t, t+h]$. In practice, perfect hedging is not possible due to the presence of transaction costs and the fact that trading cannot be done continuously. Sometimes, banks do not hedge due to the incentive to earn higher expected returns. Hence, it is of practical interest to evaluate the risk of a partially hedged, (or discretely hedged), position, or the risk inherent from an unhedged position. First, we define the future net loss of the portfolio over $[t, t+h]$ as $\exp(rh)V(t, T, S_t, X_t) - V(t+h, T, S_{t+h}, X_{t+h})$, denoted as $\Delta V_{t,h}$. Let P denote a family of probability measures $\tilde{\mathcal{P}} \sim \mathcal{P}$. Then, the GSE for the option V with respect to the family of probability measures P is defined as:

$$\rho_P\big(\Delta V_{t,h}|\mathcal{G}_u\big) = \sup_{\tilde{\mathcal{P}}\in P} E_{\tilde{\mathcal{P}}}\left[\exp\Big(-r(t+h-u)\Big)\Delta V_{t,h}\Big|\mathcal{G}_u\right], \tag{3.1}$$

where the supremum is taken over the family of probability measures P. We shall specify P explicitly in the sequel.

In order to specify the GSE, we need to specify the family P of probability measures equivalent to $\mathcal{P}$. There are different ways to specify the family P. Here, we shall specify the family P using a family of real-world probability measures over different regimes using the regime-switching Esscher transforms. Under the assumption of the Markov-modulated GBM for the price dynamics of the risky asset, this is equivalent to the change of the original appreciation rate of S to a family of regime-switching subjective appreciation rates of S, which take values in the stochastic interval governed by the values of the Markov chain process $\{X_t\}_{t\in\mathcal{T}}$. We describe this in some detail.

For each $i = 1, 2, \ldots, N$, let Λ_i be an interval $[\lambda_i^-, \lambda_i^+]$ in $\mathcal{R}$, where $\lambda_i^- \le \mu_i \le \lambda_i^+$. Define a regime-switching stochastic interval $\Lambda(X_t)$ depending on X_t such that when $X_t = e_i$, $\Lambda(e_i) = \Lambda_i$, for each $i = 1, 2, \ldots, N$. Recall that $\{\lambda_t\}_{t\in\mathcal{T}}$ is the subjective appreciation rate of the underlying stock, which switches over time according to the state of the hidden Markov chain process $\{X_t\}_{t\in\mathcal{T}}$, that is,

$$\lambda_t := \lambda(t, X_t) = \langle \lambda, X_t\rangle\,, \tag{3.2}$$

where $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_N)$ with $\lambda_i \in \Lambda_i$, for each $i = 1, 2, \ldots, N$.

Let $\theta_i^\lambda := \frac{\lambda_i - \mu_i}{\sigma_i}$, for each $i = 1, 2, \ldots, N$ and $\theta^\lambda = (\theta_1^\lambda, \theta_2^\lambda, \ldots, \theta_N^\lambda)$. Define a regime-switching Esscher parameter $\theta_t^\lambda := \big\langle\theta^\lambda, X_t\big\rangle$ associated with $\lambda_t := \langle\lambda, X_t\rangle$. Then, the regime switching Esscher transform $\mathcal{P}_{\theta^\lambda} \sim \mathcal{P}$ on $\mathcal{G}_t$ with respect to a family of parameters $\{\theta_s^\lambda\}_{s\in[0,t]}$ is given by:

$$\begin{aligned}\frac{\mathrm{d}\mathcal{P}_{\theta^\lambda}}{\mathrm{d}\mathcal{P}}\bigg|_{\mathcal{G}_t} &= \frac{\exp\left(\int_0^t \theta_s^\lambda \mathrm{d}W_s\right)}{E_{\mathcal{P}}\left[\exp\left(\int_0^t \theta_s^\lambda \mathrm{d}W_s\right)\Big|\mathcal{F}_t^X\right]}\\ &= \exp\left\{-\int_0^t \left(\frac{\mu_u - \lambda_u}{\sigma_u}\right)\mathrm{d}W_u - \frac{1}{2}\int_0^t\left(\frac{\mu_u-\lambda_u}{\sigma_u}\right)^2\mathrm{d}u\right\}, \quad t\in\mathcal{T}. \end{aligned}\tag{3.3}$$

Then, by Girsanov's theorem,

$$W_t^\lambda = W_t + \int_0^t \left(\frac{\mu_u - \lambda_u}{\sigma_u}\right)\mathrm{d}u, \quad t \in [0, T] \tag{3.4}$$

is a standard Brownian motion with respect to $\{\mathcal{G}_t\}$ under $\mathcal{P}_{\theta^\lambda}$. Hence, under $\mathcal{P}_{\theta^\lambda}$, the dynamics of the underlying stock can be rewritten as:

$$\mathrm{d}S_t = \lambda_t S_t \,\mathrm{d}t + \sigma_t S_t \,\mathrm{d}W_t^\lambda \,. \tag{3.5}$$

Let $\Lambda(u,t+h) := \bigcup_{\tau\in[u,t+h]} \Lambda(X_\tau) \subset \bigcup_{i=1}^N \Lambda_i$, for each $u \in [t,t+h)$. Now, we let $P_{\Lambda(u,t+h)}$ denote a family of subjective probability measures $\{\mathcal{P}_{\theta^\lambda}\}_{\lambda\in\Lambda(u,t+h)}$ associated with the index set $\Lambda(u,t+h)$. Then, given $\mathcal{G}_u$, we define the GSE for the option V over $[t,t+h]$ with respect to $P_{\Lambda(u,t+h)}$ as follows:

$$\begin{aligned}&\rho_{P_{\Lambda(u,t+h)}}(\Delta V_{t,h}|\mathcal{G}_u)\\ &\quad= \sup_{\lambda\in\Lambda(u,t+h)} E_{\mathcal{P}_{\theta^\lambda}}\left[\exp\Big(-r(t+h-u)\Big)\Delta V_{t,h}\Big|\mathcal{G}_u\right], \quad u\in[t,t+h),\end{aligned} \tag{3.6}$$

where the supremum is taken over the set $\Lambda(u,t+h)$.

Note that $\rho_{P_{\Lambda(u,t+h)}}$ is the GSE for the option position V, which was established at time t and will be liquidated at time $t+h$, evaluated based on the information $\mathcal{G}_u$ generated by the values of the price process of the risky asset and the Markov chain process. Note that the GSE $\rho_{P_{\Lambda(u,t+h)}}$ is coherent, (see Artzner et al. 1999).

Since (S_t, X_t) is a Markov process with respect to $\mathcal{G}_t$,

$$\begin{aligned}&E_{\mathcal{P}_{\theta^\lambda}}\left[\exp\Big(-r(t+h-u)\Big)\Delta V_{t,h}\Big|\mathcal{G}_u\right]\\ &\quad= E_{\mathcal{P}_{\theta^\lambda}}\left[\exp\Big(-r(t+h-u)\Big)\Delta V_{t,h}\Big|S_u, X_u\right].\end{aligned} \tag{3.7}$$

Hence, the GSE $\rho_{P_{\Lambda(t,t+h)}}$ can be written as a function $H(S_u, X_u, u)$ depending on S_u, X_u and u. Write $\mathbf{H}(S_u,u) := (H(S_u,e_1,u), H(S_u,e_2,u), \ldots, H(S_u,e_N,u))$. Then,

$$H(S_u, X_u, u) = \langle \mathbf{H}(S_u,u), X_u\rangle \,. \tag{3.8}$$

The following proposition presents the regime switching PDE satisfied by the GSE $H(S_u, X_u, u)$.

Proposition 3.1 *Let $\lambda^+ := (\lambda_1^+, \lambda_2^+, \ldots, \lambda_N^+)$ and $\lambda^- := (\lambda_1^-, \lambda_2^-, \ldots, \lambda_N^-)$. Suppose $K = -S\frac{\partial H}{\partial S}$ and $\lambda(K,X) = \begin{cases}\langle\lambda^+, X\rangle = \sum_{i=1}^N \lambda_i^+ I_{\{X=e_i\}} \text{ if } K<0\\ \langle\lambda^-, X\rangle = \sum_{i=1}^N \lambda_i^- I_{\{X=e_i\}} \text{ if } K>0.\end{cases}$*

Then, $H(S_u, X_u, u)$ satisfies the following regime switching PDE:

$$\frac{\partial H}{\partial u} + \frac{1}{2}\sigma_u^2 S^2 \frac{\partial^2 H}{\partial S^2} + \lambda(K,X) S\frac{\partial H}{\partial S} - rH + \langle \mathbf{H}, AX\rangle = 0, \tag{3.9}$$

with the following terminal condition:

$$H(S_{t+h}, X_{t+h}, t+h) = \exp(rh)V(t, T, S_t, X_t) - V(t+h, T, S_{t+h}, X_{t+h}).$$

Proof We shall prove the result by formulating the problem of evaluating the GSE as a Markov-switching stochastic optimal control with the Markov control variable $\lambda_t = \langle \lambda, X_t \rangle$ taking values in $\Lambda(X_t)$.

First, we notice that the controlled dynamics of S_u^λ with respect to λ can be written as follows:

$$\mathrm{d}S_u = \lambda_u S_u \mathrm{d}u + \sigma_u S_u \mathrm{d}W_u^\lambda, \quad u \in [t, t+h]. \tag{3.10}$$

Define $J^\lambda(S, X, u)$, for each $\lambda \in \Lambda(u, t+h)$ and $u \in [t, t+h]$, and $\tilde{H}(S, X, u)$ as follows:

$$J^\lambda(S, X, u) := E_{\mathcal{P}_{\theta^\lambda}}(\Delta V_{t,h} | S, X, u), \tag{3.11}$$

and

$$\tilde{H}(S, X, u) := \sup_{\lambda \in \Lambda(u, t+h)} J^\lambda(S, X, u). \tag{3.12}$$

Write $\tilde{\mathbf{H}}(S, u) := (\tilde{H}(S, e_1, u), \tilde{H}(S, e_2, u), \ldots, \tilde{H}(S, e_N, u))$. Then,

$$\tilde{H}(S, X, u) = \left\langle \tilde{\mathbf{H}}(S, u), X_u \right\rangle. \tag{3.13}$$

Then, $\tilde{H}$ satisfies the following regime-switching HJB equation:

$$\sup_{\lambda_u \in \Lambda(X_u)} \left(\frac{\partial \tilde{H}}{\partial u} + \frac{1}{2}\sigma_u^2 S^2 \frac{\partial^2 \tilde{H}}{\partial S^2} + \lambda_u S \frac{\partial \tilde{H}}{\partial S} + \left\langle \tilde{\mathbf{H}}, A X_u \right\rangle \right) = 0 \tag{3.14}$$

with the following terminal condition:

$$\tilde{H}(S_{t+h}, X_{t+h}, t+h) = \exp(rh)V(t, T, S_t, X_t) - V(t+h, T, S_{t+h}, X_{t+h}).$$

Let

$$\eta(\lambda_u) := \frac{\partial \tilde{H}}{\partial u} + \frac{1}{2}\sigma_u^2 S^2 \frac{\partial^2 \tilde{H}}{\partial S^2} + \lambda_u S \frac{\partial \tilde{H}}{\partial S} + \left\langle \tilde{\mathbf{H}}, A X_u \right\rangle. \tag{3.15}$$

Then,

$$\frac{\mathrm{d}\eta(\lambda_u)}{\mathrm{d}\lambda_u} = S \frac{\partial \tilde{H}}{\partial S}. \tag{3.16}$$

We consider the following two cases:
Case I: $S\frac{\partial \tilde{H}}{\partial S} > 0$ (i.e. $K = -S\frac{\partial \tilde{H}}{\partial S} < 0$)
In this case,

$$\frac{\mathrm{d}\eta(\lambda_u)}{\mathrm{d}\lambda_u} > 0. \tag{3.17}$$

Hence, $\eta(\lambda_u)$ is an increasing function of λ_u. This implies that $\eta(\lambda_u)$ attains its maximum when

$$\lambda_u = \left\langle \lambda^+, X_u \right\rangle = \sum_{i=1}^{N} \lambda_i^+ I_{\{X_u = e_i\}}. \tag{3.18}$$

Case II: $S\frac{\partial \tilde{H}}{\partial S} < 0$ (i.e. $K = -S\frac{\partial \tilde{H}}{\partial S} > 0$)
In this case,

$$\frac{d\eta(\lambda_u)}{d\lambda_u} < 0. \tag{3.19}$$

Hence, $\eta(\lambda_u)$ is a decreasing function of λ_u. This implies that $\eta(\lambda_u)$ attains its maximum when

$$\lambda_u = \left\langle \lambda^-, X_u \right\rangle = \sum_{i=1}^{N} \lambda_i^- I_{\{X_u = e_i\}}. \tag{3.20}$$

Then, $\tilde{H}$ satisfies the following regime-switching PDE equation:

$$\frac{\partial \tilde{H}}{\partial u} + \frac{1}{2}\sigma_u^2 S^2 \frac{\partial^2 \tilde{H}}{\partial S^2} + \lambda(K, X) S \frac{\partial \tilde{H}}{\partial S} + \left\langle \tilde{\mathbf{H}}, AX \right\rangle = 0, \tag{3.21}$$

with the following terminal condition:

$$\tilde{H}(S_{t+h}, X_{t+h}, t+h) = \exp(rh) V(t, T, S_t, X_t) - V(t+h, T, S_{t+h}, X_{t+h}).$$

Now, we notice that $H(S_u, X_u, u) = \mathrm{e}^{-r(t+h-u)} \tilde{H}(S_u, X_u, u)$.
Hence, the result follows. □

The terminal condition $H(S_{t+h}, X_{t+h}, t+h)$ can be obtained from the solution V of the regime-switching Black–Scholes P.D.E. associated with the appropriate terminal condition. With $H_i := H(S_u, e_i, u)$ $(i = 1, 2, \ldots, N)$ and $\mathbf{H} := (H_1, H_2, \ldots, H_N)$, we have $H(S_u, X_u, u) = \langle \mathbf{H}, X_u \rangle$. The following corollary presents a system of N-coupled PDEs for $\mathbf{H}$.

Corollary 3.2 *Let $K_i := -S\frac{\partial H_i}{\partial S}$, for each $i = 1, 2, \ldots, N$. Then, $\mathbf{H}$ satisfies the following system of N-coupled PDEs:*

$$\frac{\partial H_i}{\partial u} + \frac{1}{2}\sigma_i^2 S^2 \frac{\partial^2 H_i}{\partial S^2} + \lambda(K_i, e_i) S \frac{\partial H_i}{\partial S} - rH_i + \langle \mathbf{H}, Ae_i \rangle = 0, \tag{3.22}$$

with the following terminal condition:

$$H(S_{t+h}, e_i, t+h) = e^{rh} V(t, T, S_t, X_t) - V(t+h, T, S_{t+h}, e_i), \quad i = 1, 2, \ldots, N.$$

Proof The result follows from modifying the results in Proposition 3.1 and in Buffington and Elliott (2002a,b). □

In the sequel, we shall discuss the use of the GSE to evaluate the jump risk due to the hidden Markov chain. Recall from Sect. 2 that the instantaneous jump risk due to the hidden Markov chain under the measure $\mathcal{P}$ is given by:

$$\phi_1(\mu_t - r) S \mathrm{d}t - \langle \mathbf{V}, \mathrm{d}M_t \rangle \, . \tag{3.23}$$

Under $\mathcal{P}$, the jump risk $\Delta J_{t,h}$ due to the hidden Markov chain under $\mathcal{P}$ is:

$$\Delta J_{t,h} = \int_t^{t+h} \phi_1(\mu_u - r) S_u \mathrm{d}u - \int_t^{t+h} \langle \mathbf{V}, \mathrm{d}M_u \rangle \, . \tag{3.24}$$

Note that the instantaneous jump risk due to the hidden Markov chain under the measure $\mathcal{P}_{\theta_\lambda}$ is:

$$\phi_1(\lambda_t - r) S \mathrm{d}t - \langle \mathbf{V}, \mathrm{d}M_t \rangle \, . \tag{3.25}$$

Hence, under $\mathcal{P}_{\theta_\lambda}$,

$$\Delta J_{t,h} = \int_t^{t+h} \phi_1(\lambda_u - r) S_u \mathrm{d}u - \int_t^{t+h} \langle \mathbf{V}, \mathrm{d}M_u \rangle \, . \tag{3.26}$$

Then, given $\mathcal{G}_u$, we define the GSE for $\Delta J_{t,h}$ with respect to $P_{\Lambda(u,t+h)}$ as follows:

$$\begin{aligned} &\rho_{P_{\Lambda(u,t+h)}}\left(\Delta J_{t,h} | \mathcal{G}_u\right) \\ &= \sup_{\lambda \in \Lambda(u,t+h)} E_{\mathcal{P}_{\theta\lambda}}\left[\exp\Big(-r(t+h-u)\Big)\Delta J_{t,h}\Big|\mathcal{G}_u\right], \quad u \in [t, t+h), \end{aligned} \tag{3.27}$$

Since $E_{\mathcal{P}_{\theta\lambda}}[\int_t^{t+h} \langle \mathbf{V}, dM_u \rangle \,|\mathcal{G}_u] = 0$,

$$\rho_{P_{\Lambda(u,t+h)}}\big(\Delta J_{t,h}|\mathcal{G}_u\big)$$
$$= \sup_{\lambda \in \Lambda(u,t+h)} E_{\mathcal{P}_{\theta\lambda}}\left[\exp\Big(-r(t+h-u)\Big)\int_t^{t+h} \phi_1(\lambda_u - r) S_u \mathrm{d}u \,\Big|\, \mathcal{G}_u\right]. \quad (3.28)$$

Now, for each $u \in [t, t+h]$ let $I_u := \int_t^u \phi(v)\lambda_v S_v dv$ and $O_u := \int_t^u \phi(v) S_v dv$. Write $H^J(S, X, I, O, u)$ for the GSE $\rho_{P_{\Lambda(u,t+h)}}\big(\Delta J_{t,h}|\mathcal{G}_u\big)$. Note that

$$\frac{\mathrm{d}I}{\mathrm{d}u} = \phi(u)\lambda_u S_u, \quad \frac{\mathrm{d}O}{\mathrm{d}u} = \phi(u) S_u. \quad (3.29)$$

With $H_i^J := H^J(S, e_i, I, O, u)$ and $\mathbf{H}^J := (H_1^J, H_2^J, \ldots, H_N^J)$, we have $H^J(S, X, I, O, u) = \langle \mathbf{H}^J, X_u \rangle$. Then, $\mathbf{H}^J$ satisfies the following system of N-coupled PDEs:

$$\frac{\partial H_i^J}{\partial t} + \phi(u)\tilde{\lambda}(\tilde{K}_i, e_i) S_u \frac{\partial H_i^J}{\partial I} + \phi(u) S_u \frac{\partial H_i^J}{\partial O} + \frac{1}{2}\sigma_i^2 S_u^2 \frac{\partial^2 H_i^J}{\partial S^2} + \lambda(K_i, e_i) S_u \frac{\partial H_i^J}{\partial S}$$
$$- rH_i^J + \Big\langle \mathbf{H}^J, A e_i \Big\rangle = 0, \quad i = 1, 2, \ldots, N, \quad (3.30)$$

where $\lambda(K_i, e_i)$ is defined as before, $\tilde{K}_i$ and $\tilde{\lambda}(\tilde{K}_i, e_i)$ are defined as follows: $\tilde{K}_i := -\phi(u) S \frac{\partial H_i^J}{\partial I}$ and $\tilde{\lambda}(\tilde{K}_i, e_i) = \begin{cases} \lambda_i^+ \text{ if } \tilde{K}_i < 0 \\ \lambda_i^- \text{ if } \tilde{K}_i > 0. \end{cases}$
The terminal condition is then given by:

$$H^J(S, e_i, I, O, t+h) = \sup_{\lambda \in \Lambda(t,t+h)} (I_{t+h} - rO_{t+h})$$
$$= \hat{I}_{t+h} - rO_{t+h}, \quad i = 1, 2, \ldots, N, \quad (3.31)$$

where $\hat{I}_{t+h} := \int_t^{t+h} \phi_1(u)(\hat{\lambda}(\phi_1(u), X_u) - r) S_u du$ with $\hat{\lambda}(\phi_1(u), X_u)$ given by:
$\hat{\lambda}(\phi(u), X_u) = \begin{cases} \langle \lambda^+, X_u \rangle = \sum_{i=1}^N \lambda_i^+ I_{\{X_u = e_i\}} \text{ if } \phi(u) > 0 \\ \langle \lambda^-, X_u \rangle = \sum_{i=1}^N \lambda_i^- I_{\{X_u = e_i\}} \text{ if } \phi(u) < 0. \end{cases}$

4 European-style barrier options

In this section, we shall adopt the PDE approach to calculate the generalized scenarios expectation (GSE) for European-style barrier options. In particular, we consider a single down-and-out European call option with expiry time T. Let L and K be the barrier level and strike price, respectively, of the down-and-out option. We only consider the case where $K > L$ and assume that $S_0 > L$. Let $bd_0(S, X, t)$ denote the price of the down-and-out option at time t given

$S_t = S$ and $X_t = X$ under the assumption that the call has not been knocked out prior to time t. Then, by employing the risk-neutral regime-switching Esscher transform $\mathcal{Q}_{\tilde{\theta}}$ described in Sect. 3, $bd_0(S, X, t)$ is given by:

$$\begin{aligned} & bd_0(S, X, t) \\ & \quad = E_{\mathcal{Q}_{\tilde{\theta}}}[\mathrm{e}^{-r(T-t)} \max\{S_T - K, 0\} I_{\{\min_{t \le t' \le T} S_{t'} \ge L\}} | S_t = S, X_t = X]. \end{aligned} \tag{4.1}$$

Let $bd_{0i} := bd_0(S, e_i, t)$ and $\mathbf{bd_0} := (bd_{01}, bd_{02}, \ldots, bd_{0N})$. Then, $bd_0(S, X, t) = \langle \mathbf{bd_0}, X_t \rangle$ and $\mathbf{bd_0}$ satisfies the following system of N-coupled P.D.E.s:

$$- rbd_{0i} + \frac{\partial bd_{0i}}{\partial t} + rS\frac{\partial bd_{0i}}{\partial S} + \frac{1}{2}\sigma_i^2 S^2 \frac{\partial^2 bd_{0i}}{\partial S^2} + \langle \mathbf{bd_0}, Ae_i \rangle = 0, \tag{4.2}$$

where the boundary and terminal conditions for the down-and-out option are given as follows:

$$bd_0(L, e_i, t) = 0, \quad bd_0(S, e_i, t) \sim S \quad \text{as } e^{-r(T-t)} S \to \infty$$

and

$$bd_0(S_T, e_i, T) = \max\{S_T - K, 0\} I_{\{\min_{0 \le t \le T} S_t \ge L\}}, \quad i = 1, 2, \ldots, N.$$

For the fixed time $t \ge 0$, we assume that the down-and-out call option has not yet extinguished at time t. As in section two, we focus on an unhedged position of the down-and-out call option from the buyer's viewpoint. Since we decide to measure the risk of the naked position during the time interval $[t, t+h]$, we assume that the naked position will not be liquidated in secondary markets during the time horizon $[t, t+h]$. Given the market information $\mathcal{G}_u$ up to time $u \in [t, t+h)$, we define a risk measure for the naked position over $[t, t+h]$. First, we need to distinguish two cases, say $\min_{0 \le v \le u} S_v \le L$ and $\min_{0 \le v \le u} S_v > L$ for each fixed $u \in [t, t+h)$. If $\min_{0 \le v \le u} S_v \le L$, the down-and-out option becomes worthless at time u and so the GSE equals $e^{r(u-t)} bd_0(S_t, X_t, t)$ at time u. If $\min_{0 \le v \le u} S_v > L$, the GSE at time u is defined as follows:

$$\rho_{P_{\Lambda(u,t+h)}}\big(\Delta\, bd_0(t,h) \mid \mathcal{G}_u\big) = \sup\big\{-E_{\mathcal{P}_\lambda}\big(\mathrm{e}^{-r(t+h-u)} \Delta\, bd_0(t,h) | \mathcal{G}_u\big) \big| \lambda \in \Lambda(u, t+h)\big\}, \tag{4.3}$$

where $\Delta\, bd_0(t,h) = bd_0(S_{t+h}, X_{t+h}, t+h) I\big\{\min_{u \le u' \le t+h} S_{u'} > L\big\} - \mathrm{e}^{rh} bd_0(S_t, X_t, t)$ and $I(A)$ is an indicator function of the event A.

Since (S_t, X_t) is a two-dimensional Markov process with respect to $\mathcal{G}_t$, the GSE for the down-and-out option can be written as prior to knock out a function $H^b(S_u, X_u, u)$ depending on S_u, X_u and u. Then, as in Sect. 3, it can be shown that $H^b(S, X, u)$ satisfies the regime-switching PDE in Proposition 3.1 when $S > L$ associated with the following boundary and terminal conditions:

$$H^b(L, X_u, u) = \mathrm{e}^{r(u-t)} bd_0(S_t, X_t, t)$$

and

$$H^b(S_{t+h}, X_{t+h}, t+h) = \mathrm{e}^{rh} bd_0(S_t, X_t, t) - bd_0(S_{t+h}, X_{t+h}, t+h)$$
$$I\left\{\min_{t \le u \le t+h} S_u > L\right\},$$

Now, write $H_i^b := H^b(S_u, e_i, u)$ ($i = 1, 2, \dots, N$) and $\mathbf{H}^b := (H_1^b, H_2^b, \dots, H_N^b)$. Then, $H^b(S_u, X_u, u) = \langle \mathbf{H}^b, X_u \rangle$. Then, $\mathbf{H}^b$ satisfies the following system of N-coupled PDEs in Corollary 3.2 when $S > L$ associated with the following boundary and terminal conditions:

$$H^b(L, e_i, u) = \mathrm{e}^{r(u-t)} bd_0(S_t, X_t, t)$$

and

$$H^b(S_{t+h}, e_i, t+h) = \mathrm{e}^{rh} bd_0(S_t, X_t, t) - bd_0(S_{t+h}, e_i, t+h) I\left\{\min_{t \le u \le t+h} S_u > L\right\},$$

where $i = 1, 2, \dots, N$.

5 American-style options

In this section, as in Siu and Yang (2000), we shall first define the GSE for a single American-style contract with an arbitrary payoff function and expiry time T to incorporate the feature of early exercise. Then, we shall derive a regime-switching partial differential system governing the dynamics of the GSE of the American-style contract. The regime-switching partial differential system can be regarded as a regime-switching free-boundary problem, which has been discussed in Buffington and Elliott (2002a,b).

Let V^a be the value of an American-style derivative at time t with expiry time T. We can write V^a as a function $V^a(S_t, X_t, t)$ depending on S_t, X_t and t. Let $V_i^a := V^a(S, e_i, t)$ and $\mathbf{V}^a := (V_1^a, V_2^a, \dots, V_N^a)$. Then, $V^a(S, X, t) = \langle \mathbf{V}^a, X \rangle$. Following the analysis in Buffington and Elliott (2002a,b), $\mathbf{V}^a$ satisfies the following system of N-coupled partial differential systems:

$$\frac{\partial V_i^a}{\partial t} + \frac{1}{2}\sigma_i^2 S^2 \frac{\partial^2 V_i^a}{\partial S^2} + r S \frac{\partial V_i^a}{\partial S} - r V_i^a + \langle \mathbf{V}^a, A e_i \rangle \le 0,$$
$$V^a(S_t, e_i, t) \ge f(S_t, t),$$

where $f : \mathcal{R}^+ \times [0, T] \to \mathcal{R}$ is a continuous reward function satisfying the linear growth condition $|f(s,t)| \le c_1 + c_2 s$ for some real constants c_1, c_2;

$$\left(\frac{\partial V_i^a}{\partial t} + \frac{1}{2}\sigma_i^2 S^2 \frac{\partial^2 V_i^a}{\partial S^2} + r S \frac{\partial V_i^a}{\partial S} - r V_i^a + \langle \mathbf{V}^a, A e_i \rangle\right)(V_i^a - f) = 0,$$

and

$$V^a(S_T, e_i, T) = f(S_T, T), \quad i = 1, 2, \ldots, N.$$

We suppose that the American option will not be liquidated during the time horizon $[t, t+h]$ and that it is an unhedged position. However, as in Siu and Yang (2000), we allow the possibility of early exercise at any intermediate time $u \in [t, t+h]$. Let τ^* be the optimal stopping time for the above optimal stopping problem for the American option. We first assume that $\tau^* > t$ (that is, the American option has not yet been exercised at time t) for a fixed time $t \geq 0$. Given the market information $\mathcal{G}_u$ up to time $u \in [t, t+h)$, we define the GSE for the naked position of the American option over $[u, t+h]$ with the possibility that the American option may be exercised optimally during the time horizon $[u, t+h]$. Here, we need to distinguish two cases (namely $t < \tau^* \leq u$ and $\tau^* > u$) for each $u \in [t, t+h)$. If $t < \tau^* \leq u$, the American option is exercised at some time $u' \in (t, u]$, and hence it is trivial that the GSE equals $e^{r(u-u')} f(S_{u'}, u') - e^{r(u-t)} V^a(S_t, X_t, t)$. Following the notations in Sect. 3, if $\tau^* > u$, the GSE is defined in the following expectation form:

$$\begin{aligned} H^a(S_u, X_u, u) = \sup\{ &-E_{\mathcal{P}_\lambda}[\mathrm{e}^{-r(J_u^{t+h}(\tau^*)-u)} V^a(S_{J_u^{t+h}(\tau^*)}, X_{J_u^{t+h}(\tau^*)}, J_u^{t+h}(\tau^*)) | \mathcal{G}_u] \\ &| \lambda \in \Lambda(u, t+h)\} + \mathrm{e}^{r(u-t)} V^a(S_t, X_t, t), \end{aligned} \tag{5.1}$$

where the function J_u^{t+h} of the optimal stopping time τ^* is defined as follows:

$$J_u^{t+h}(\tau^*) = \begin{cases} \tau^* & \text{if } \tau^* \in [u, t+h) \\ t+h & \text{if } \tau^* \in [t+h, T] \end{cases}$$

and the function V^a is the price process of the American derivative.

By introducing the function J_u^{t+h} of the optimal stopping time τ^*, we can incorporate the feature of early exercise during the time horizon $[u, t+h]$ into the calculation of the GSE at time u. Now, write $H_i^a := H^a(S_u, e_i, u)$ $(i = 1, 2, \ldots, N)$ and $\mathbf{H}^a := (H_1^a, H_2^a, \ldots, H_N^a)$. Then, $H^a(S_u, X_u, u) = \langle \mathbf{H}^a, X_u \rangle$. Then, $\mathbf{H}^a$ satisfies the following system of N-coupled partial differential equations:

$$\begin{aligned} &\frac{\partial H_i^a}{\partial u} + \frac{1}{2}\sigma_i^2 S^2 \frac{\partial^2 H_i^a}{\partial S^2} + \lambda(K_i^a, e_i) S \frac{\partial H_i^a}{\partial S} - r H_i^a + \langle \mathbf{H}^a, A e_i \rangle \geq 0, \\ &H^a(S_u, e_i, u) \leq -f(S_u, u) + \mathrm{e}^{r(u-t)} V^a(S_t, X_t, t), \\ &H^a(S_{t+h}, e_i, t+h) = \mathrm{e}^{rh} V^a(S_t, X_t, t) - V^a(S_{t+h}, e_i, t+h), \\ &\left(\frac{\partial H_i^a}{\partial u} + \frac{1}{2}\sigma_i^2 S^2 \frac{\partial^2 H_i^a}{\partial S^2} + \lambda(K_i^a, e_i) S \frac{\partial H_i^a}{\partial S} - r H^a + \langle \mathbf{H}^a, A e_i \rangle \right) (H_i^a - \tilde{f}) = 0, \end{aligned}$$

where

$$K_i^a = -S\frac{\partial H_i^a}{\partial S},$$
$$\lambda(K_i^a, e_i) = \begin{cases} \lambda_i^+ & \text{if } K_i^a < 0 \\ \lambda_i^- & \text{if } K_i^a > 0 \end{cases}$$

and

$$\tilde{f}(S_u, u) = -f(S_u, u) + e^{r(u-t)} V^a(S_t, X_t, t),$$

where $i = 1, 2, \ldots, N$.

6 Conclusion and further research

We have considered a PDE approach for evaluating coherent risk measures for derivative instruments in a Markov-modulated Black–Scholes environment. The PDE approach can provide market practitioners with a flexible and effective way to evaluate risk measures in the Markov-modulated Black–Scholes model. It is especially useful for the case of American options, in which the possibility of early exercise needs to be taken into consideration in both the pricing and risk measurement parts. We have considered the use of the regime-switching Esscher transform and the delta-neutral hedging for pricing and hedging options in the Markov-modulated Black-Scholes market. We have derived the PDEs satisfied by the GSE for European-style options and American-style options. We have demonstrated the use of the PDE approach for evaluating the GSE for a down-and-out barrier option.

For further research, numerical and computational problems represent interesting topics. Following the numerical methods in Yao et al. (2006), one can develop some accurate and efficient numerical schemes for the regime-switching PDE framework.

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